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A Numerical Method with Floating Meshes for Singularly Perturbed Problems with a Concentrated Disturbance in the Initial Data[1]

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We consider the parabolic Dirichlet problem on an interval for a singularly perturbed parabolic equation. The initial conditions of the problem have a local disturbance over a small area of width 2δ. The perturbation parameters ε^2, that is the coefficient multiplying the highest derivative, and δ can take arbitrary values from the half-open intervals $(0,1]$ and $(0,d]$, respectively, where d is the width of the segment. As $\varepsilon = 0$, the parabolic equation degenerates to the first-order hyperbolic equation, which contains the derivatives with respect to the space and time variables. Using the method of additive splitting of the singularities, and floating condensed meshes with the nodes located along the characteristics of the reduced equation, we construct difference schemes that are uniformly convergent with respect to the parameters ε and δ. We restrict ourselves to the case, where the disturbance trace does not cross the boundary of the domain.

1. Introduction

The solutions of boundary value problems for PDEs, in particular for singularly perturbed equations, have a restricted smoothness if there is a local disturbance

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of the initial conditions Smooth initial conditions vary by a finite value over a narrow subdomain. The solutions of such problems have high-order derivatives which increase unboundedly when the value δ (i.e., the half-width of the subdomain) and/or the perturbation parameter ε (ε^2 is the coefficient multiplying the highest derivative) tend to zero. The presence of the local disturbances results in interior (transient) layers appearing for the small values of the parameters ε and δ.

The restricted smoothness of the solutions of such boundary value problems may give rise to difficulties in the numerical solution (see, for example, [1]–[4]). For small values of the parameters ε and δ, the errors in the approximate solutions obtained by classical difference schemes may become commensurable with the solution of the boundary value problem itself (see, e.g., Theorem 1). Therefore, there is an interest to construct special difference schemes for which the solution error is independent of the value of the parameters ε and δ, i.e., difference schemes that converge uniformly with respect to the parameters ε and δ (converge (ε, δ)-uniformly).

In this paper we consider the boundary value problem for singularly perturbed parabolic equations on a segment. The initial conditions of the problem have a local disturbance: they vary by a finite value in the small interval of the local disturbance, which is a subdomain with half-width δ.

The reduced equation (the equation for $\varepsilon = 0$) contains the first-order derivatives with respect to the space and time variables. For such boundary value problems we construct special difference schemes whose solutions converge to the solution of the boundary value problems (ε, δ)-uniformly.

To solve the boundary value problem for small values of the parameters ε and δ, we use the additive representation of the solution (cf. [5]), where the term with the singular part of the solution, generated by the local disturbance, is obtained separately. For the construction of the discrete equations we use classical difference approximations of the boundary value problems [6]. For the computation of the smooth part of the solution and also for the solution of the problem with values of the parameters that are not too small, we use uniform rectangular grids. For the computation of the singular part of the solution, we use both rectangular grids and floating "characteristic" grids, of which the nodes are placed along the "trace" of the local disturbance in the initial conditions, that is, along the characteristic curve passing through the centre of the interval of the local disturbance. When the reduced equation does not contain the first order space derivative, we construct a special rectangular piecewise uniform grid (in space and in time), which condenses in the neighbourhood of the local disturbance. On such a grid the difference scheme converges (ε, δ)-uniformly.

In the case that the reduced equation contains a first-order derivative in space, the special piecewise uniform characteristic meshes allow us to find (with (ε, δ)-uniform accuracy) the surface along which the interval with the local disturbance moves. In short, the distance between the numerical and exact traces of the disturbance tends to zero (ε, δ)-uniformly as the number of nodes grows. For small values of the parameter δ, the derivatives of the solution of the boundary value problem can take arbitrarily large values. Due to this fact, on a simple uniform grid, the approximate solution of the boundary value problem does not converge to the exact solution (ε, δ)-uniformly in the discrete maximum norm. However, on the local grid the point-wise error of the trace of the disturbance tends to zero, (ε, δ)-uniformly in the discrete maximum norm. We shall call such approximations convergent (ε, δ)-uniformly up to shift of the distance between the numerical and the exact traces of the disturbance (or, in short, they converge (ε, δ)-uniformly up to the shift).

2. Setting the problem. Problem formulation

2.1. On the interval $D = \{x : \ -d < x < d\}$ we consider Dirichlet's problem[2] for the parabolic equation

$$L_{(2.1)} u(x,t) = f(x,t), \ (x,t) \in G, \qquad u(x,t) = \varphi(x,t), \ (x,t) \in S. \qquad (2.1)$$

Here $G = D \times (0,T]$, $S = S(G) = \overline{G} \setminus G$,

$$\begin{aligned} L_{(2.1)} \ &\equiv \ \varepsilon^2 \left\{ a(x,t) \frac{\partial^2}{\partial x^2} + b(x,t) \frac{\partial}{\partial x} \right\} - c(x,t) - p(x,t) \frac{\partial}{\partial t} \\ &\equiv \ \varepsilon^2 L^2_{(2.1)} + L^1_{(2.1)}, \end{aligned}$$

the functions $a(x,t)$, $b(x,t)$, $c(x,t)$, $p(x,t)$, $f(x,t)$, and also $\varphi(x,t)$ are sufficiently smooth and bounded, on the set $\overline{G}$ and on the lateral sides of the set G, respectively. The boundary function $\varphi(x,t)$ is continuous on S. The coefficients satisfy the conditions:

$$a_0 \le a(x,t) \le a^0, \ \ (x,t) \in \overline{G},$$

and $p(x,t) \ge p_0$, $c(x,t) \ge 0$, $(x,t) \in G$, $a_0, p_0 > 0$. The parameter ε takes arbitrary values from the half-open interval (0,1].

In this paper, the initial data can be considered as being the sum of a continuous function and a finite bounded disturbance with small support. The function $\varphi(x,t)$ depends on the parameter δ which takes any value from the half-open interval $(0,d]$. The function $\varphi(x,t) = \varphi(x,t;\delta)$ for $t = 0$, is smooth and bounded on the set $\overline{D}$, but changes by a finite value in the δ-neighbourhood of the origin $\Gamma_0 = \{x : \ x = 0\} \subset D$. We describe the function $\varphi(x,t)$, $(x,t) \in S$ more precisely. Let $S_0 = \{(x,t) : \ x \in \overline{D}, \ t = 0\}$ be the lower boundary of the set G and let $S^L = S \setminus S_0$. Let $\varphi^+(x)$, $x \in D^+$, $\varphi^-(x)$, $x \in D^-$, where $D^+ = (0,d)$, $D^- = (-d,0)$, be sufficiently smooth functions, and assume $\varphi(x,t) = \varphi_0(x)$, $(x,t) \in S_0$. The function $\varphi_0(x)$ coincides with one of the functions $\varphi^+(x)$, $\varphi^-(x)$ for $|x| \ge \delta$ and smoothly connects these functions in the region $|x| \le \delta$. Moreover, the function $\varphi(x,t)$ satisfies the estimates

[2] The notation $L_{(j.k)}$ $(M_{(j.k)}, \ f_{(j.k)}(x,t))$ means that these operators (or constants, functions) are first introduced in the formula $(j.k)$.

$$\left|\frac{\partial^{k_0}}{\partial t^{k_0}}\varphi(x,t)\right| \le M, \quad (x,t)\in S^L; \quad \left|\frac{\partial^{k}}{\partial x^{k}}\varphi(x,t)\right| \le M, \quad |x|\ge \delta, \tag{2.2a}$$

$$\left|\frac{\partial^{k}}{\partial x^{k}}\varphi(x,t)\right| \le M\delta^{-k}, \quad |x|<\delta, \quad (x,t)\in S_0.$$

Note that, generally speaking, $\varphi^+(x) \not\equiv \varphi^-(x)$, and $\max_x |\varphi_0(x)-\varphi^{\pm}(x)| = \mathcal{O}(1)$, $x\in \Gamma_0$. Problems of this type arise, e.g., when heat transfer processes are modelled (for $t \ge t_0 > 0$, $t_0 = \varepsilon^{-2}\delta^2$) in the case of concentrated instantaneous sources [8].

We will consider the special case of problem (2.1), where the functions $\varphi^+(x)$, $\varphi^-(x)$ satisfy the additional condition

$$\varphi^+(x) = \varphi^-(x), \; x\in\Gamma_0. \tag{2.2b}$$

The solution of the problem is regarded as a function $u \in C^{2,1}(G)\cap C(\overline{G})$ which is bounded on $\overline{G}$ and satisfies the differential equation on G, and the boundary condition on S. Assume that the compatibility conditions [7] which ensure the smoothness of the solution for each set of the values of the parameters ε and δ are fulfilled on the set $\{(x,t):\; x\in \overline{D}\setminus D$.

As the parameter ε tends to zero, boundary layers appear in a neighbourhood of the set S^L, and a transient layer appear in a neighbourhood of the set $S^* = \Gamma_0 \times (0,T]$. Note that problem (2.1) is singular even for $\varepsilon = 1$; the smoothness of the solution is deteriorated when the parameter δ tends to zero. This means that the solution of the problem is sufficiently smooth for a fixed value of the parameter δ, but its derivatives increase unboundedly as $\delta\to 0$.

Here and below by M, M_i (m, m_i) we denote sufficiently large (small) positive constants independent of the parameters ε and δ. In the case of discrete problems these constants do not depend on the discretisation parameters in the difference schemes used.

2.2. Together with problem (2.1), we consider the boundary value problem

$$L_{(2.3)}u(x,t) = f(x,t),\; (x,t)\in G, \quad u(x,t)=\varphi(x,t),\; (x,t)\in S, \tag{2.3}$$

where
$$\begin{aligned} L_{(2.3)} \;&\equiv\; \varepsilon^2\left\{a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x}\right\} + \\ &+ b^1(x,t)\frac{\partial}{\partial x} - c(x,t) - p(x,t)\frac{\partial}{\partial t} \equiv \varepsilon^2 L^2_{(2.1)} + L^1_{(2.3)}. \end{aligned}$$

The data of problem (2.3) satisfy the same conditions as those formulated for problem (2.1), and the functions $b(x,t)$, $b^1(x,t)$ are sufficiently smooth on $\overline{G}$.

For the singularly perturbed boundary value problems (2.1) and (2.3) we are interested in the construction of special schemes which are (ε, δ)-uniformly convergent.

For simplicity, we assume that the data of the problem satisfy conditions under which no boundary layers appear. The construction of the schemes for problems with boundary layers is studied, e.g., in [4], [9]–[12].

3. A-priori estimates for the solutions and its derivatives

Using the technique from [4], [9]–[12], in this section we find estimates for the solutions of boundary value problems (2.1) and (2.3), and for their derivatives.

3.1. In problem (2.1) we switch to a new coordinate system in which the derivatives (with respect to the space variable) of the initial function are δ-uniformly bounded. For $\varepsilon^2 \le \delta$ we pass to the variables ζ, τ, where $\zeta = \varepsilon^{-2}x$, $\tau = \varepsilon^{-2}t$, and, for $\varepsilon^2 > \delta$ to the variables χ, θ, where $\chi = \delta^{-1}x$, $\theta = \varepsilon^2\delta^{-2}t$. In these new variables the space and time derivatives are bounded (ε,δ)-uniformly. Returning to the original data, we can derive the following bounds (or estimate)

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} u(x,t)\right| \le M\delta^{-k}\left[1+\varepsilon^{2k_0}\delta^{-2k_0}\right], \quad (x,t)\in\overline{G}. \tag{3.1}$$

In this way also problem (2.3) can be investigated (see estimate (3.14a).

3.2. In the following subsections we derive a number of estimates based on the asymptotic representation of the solution. It is convenient to use the notation in the new variables for boundary value problems (2.1) and (2.3). In problem (2.3) we pass to the new variables (characteristic with respect to x) y, t:

$$y = Y(x,t), \quad x = Y(x,0), \tag{3.2a}$$

so that the operator $L^1_{(2.3)}$ takes the form

$$L^1_{(3.3)} \equiv -\widetilde{c}(y,t) - \widetilde{p}(y,t)\frac{\partial}{\partial t}. \tag{3.3}$$

Here the coefficients $v(x,t)$ and $\widetilde{v}(y,t)$ of the operators $L^1_{(2.3)}$ and $L^1_{(3.3)}$, respectively, are related by: $\widetilde{v}(y,t) = v_y(y,t) \equiv v(X(y,t),t)$, $v(x,t) \equiv \widetilde{v}_x(x,t) = \widetilde{v}(Y(x,t),t)$, where

$$x = X(y,t) \tag{3.2b}$$

is the transformation inverse to (3.2a).

In the variables y,t we arrive at the problem

$$L_{(3.4)}\widetilde{u}(y,t) = \widetilde{f}(y,t),\ (y,t)\in\widetilde{G}, \quad \widetilde{u}(y,t) = \widetilde{\varphi}(y,t),\ (y,t)\in\widetilde{S}. \tag{3.4}$$

Here $\widetilde{u}(y,t) = u_y(y,t)$, $(u(x,t) = \widetilde{u}_x(x,t))$, $\widetilde{G}^0 = G^0_y$, $\widetilde{\varphi}(y,t) = \varphi(X_1(y,t),t)$, G^0_y is the image of the set G^0, $G^0 \subseteq \overline{G}$,

$$L_{(3.4)} \equiv \varepsilon^2 L^2_{(3.4)} + L^1_{(3.3)}, \quad L^2_{(3.4)} \equiv A(y,t)\frac{\partial^2}{\partial y^2} + B(y,t)\frac{\partial}{\partial y}$$

with

$$A(y,t) = a(y,t)\left[\frac{\partial}{\partial x}Y_y(y,t)\right]^2,$$

where $Y_y(y,t) = Y(X(y,t),t)$, $(\partial/\partial x)Y_y(y,t) = (\partial/\partial x)Y(X(y,t),t)$.

Thus we reduced problem (3.4) to a problem of the form (2.1).

3.3. Assuming the condition (2.2) to be satisfied, we obtain the estimates for the regular and singular parts of the solution of problem (2.1). It is convenient to represent the function $\varphi(x,t)$ as a sum of the functions

$$\varphi(x,t) = \varphi^1(x,t) + \varphi^2(x,t),\ (x,t) \in S, \tag{3.5}$$

where the functions $\varphi^i(x,t)$ satisfy the conditions

$$\left|\frac{\partial^k}{\partial x^k}\varphi^1(x,t)\right| \le M, \quad (x,t) \in S_0, \quad k \le 1;$$

and

$$\left|\frac{\partial^k}{\partial x^k}\varphi^2(x,t)\right| \le M\delta^{-k},\ (x,t) \in S_0, \quad \varphi^2(x,t) \equiv 0,\ r(x,\Gamma_0) \ge \delta,\ (x,t) \in S,$$

where $r(x,\Gamma_0)$ is the distance from the point x to Γ_0.

For simplicity, the following conditions are assumed to hold:

$$\varphi^1(\cdot,0), \varphi^2(\cdot,0) \in C^{l+\alpha}(S_0), \quad l \ge K, \quad K \ge 6, \quad \alpha > 0, \tag{3.6a}$$
$$\left|\frac{\partial^k}{\partial x^k}\varphi^1(x,t)\right| \le M, \quad (x,t) \in S_0, \quad k \le 6. \tag{3.6b}$$

Now we represent the solution of problem (2.1) as a sum of the functions

$$u(x,t) = U(x,t) + V(x,t),\ (x,t) \in \overline{G}. \tag{3.7}$$

The functions $U(x,t), V(x,t)$, which are the smooth and singular parts of the solution of problem (2.1), are the solutions of the problems

$$L_{(2.1)}U(x,t) = f(x,t),\ (x,t) \in G, \quad U(x,t) = \varphi^1(x,t),\ (x,t) \in S; \tag{3.8a}$$
$$L_{(2.1)}V(x,t) = 0,\ (x,t) \in G, \quad V(x,t) = \varphi^2(x,t),\ (x,t) \in S. \tag{3.8b}$$

The functions $U(x,t), V(x,t)$ satisfy the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}U(x,t)\right| \le M, \quad (x,t) \in \overline{G}; \tag{3.9}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}}V(x,t)\right| \le M\delta^{-k}\left[1 + \varepsilon^{2k_0}\delta^{-2k_0}\right], \quad (x,t) \in \overline{G},\ k + 2k_0 \le 4.$$

We introduce the auxiliary function $V_0(x,t),\ (x,t) \in \overline{G}$, which is the solution of the problem

$$\begin{aligned} L_{(3.10)}V_0(x,t) \quad &\equiv \quad \left\{\varepsilon^2 a(0,t)\frac{\partial^2}{\partial x^2} - c(0,t) - p(0,t)\frac{\partial}{\partial t}\right\} V_0(x,t) = 0, \\ & \quad (x,t) \in G, \\ V_0(x,t) \quad &= \quad \varphi^2(x,t), \\ & \quad (x,t) \in S. \end{aligned} \tag{3.10}$$

The function $V_0(x,t)$ satisfies the estimates

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} V_0(x,t)\right| \;\le\; M\varepsilon^{2k_0}\delta^{-k-2k_0} \;\times \tag{3.11a}$$
$$\times\;\left(1+\varepsilon\delta^{-1}t^{1/2}\right)^{-2k_0-k-1} \exp\left(-m_2\delta^{-2}(1+\varepsilon\delta^{-1}t^{1/2})^{-2}r^2(x,\Gamma_0)\right),$$
$$\text{if} \quad \varepsilon \ge \delta;$$
$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} V_0(x,t)\right| \;\le\; M\delta^{-k}\exp(-m_1\delta^{-1}r(x,\Gamma_0)),$$
$$\text{if} \quad \varepsilon \le \delta, \quad (x,t) \in \overline{G},$$

where m_1 is an arbitrary number, m_2 is an arbitrary number satisfying the condition $m_2 < m_2^0$, $m_2^0 = 4^{-1}\min\limits_{\overline{G}}[\,a^{-1}(x,t)p(x,t)\,]$. For small values of the parameter δ the functions $V(x,t)$ and $V_0(x,t)$ are close to each other:

$$\begin{aligned} |V(x,t)-V_0(x,t)| \;&\le\; M\delta^{1-\nu}, \quad \varepsilon \ge \delta; \\ |V(x,t)-V_0(x,t)| \;&\le\; M\delta, \quad \varepsilon \le \delta, \quad (x,t) \in \overline{G}, \end{aligned} \tag{3.11b}$$

where $\nu > 0$ is an arbitrarily small number.

THEOREM 1. *Let the conditions* (2.2) *hold. Then the estimate* (3.1) *is valid for the solution of problem* (2.1). *If, besides this, the conditions* (3.5), (3.6) *are valid, then the functions* $U(x,t)$, $V(x,t)$, *that are the components of representation* (3.7), *satisfy the estimates* (3.9) *and* (3.11).

3.4. For the solution of boundary value problem (2.3) and its components from the representation

$$u(x,t) = U(x,t) + V(x,t), \quad (x,t) \in \overline{G}, \tag{3.12}$$

where $U(x,t)$, $V(x,t)$ are the solutions of the problems

$$L_{(2.3)}U(x,t) = f(x,t),\ (x,t) \in G, \quad U(x,t) = \varphi^1(x,t),\ (x,t) \in S; \tag{3.13a}$$
$$L_{(2.3)}V(x,t) = 0,\ (x,t) \in G, \quad V(x,t) = \varphi^2(x,t),\ (x,t) \in S, \tag{3.13b}$$

the following estimates are valid:

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} u(x,t)\right| \le M\delta^{-k-k_0}\left[1+\varepsilon^{2k+2k_0}\delta^{-2k-2k_0}\right], \quad (x,t) \in \overline{G}, \tag{3.14a}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} U(x,t)\right| \le M, \tag{3.14b}$$

$$\left|\frac{\partial^{k+k_0}}{\partial x^k \partial t^{k_0}} V(x,t)\right| \le M\delta^{-k-k_0}[1+\varepsilon^{2k+2k_0}\delta^{-2k-2k_0}], \quad (x,t) \in \overline{G},\ k+2k_0 \le 4.$$

We use the notation $\widetilde{V}_0(y,t)$ for the solution of the auxiliary problem

$$\begin{aligned} L_{(3.15)}\widetilde{V}_0(y,t) &\equiv \left\{\varepsilon^2 A(0,t)\frac{\partial^2}{\partial y^2} - \widetilde{c}(0,t) - \widetilde{p}(0,t)\frac{\partial}{\partial t}\right\}\widetilde{V}_0(y,t) = 0, \quad (3.15)\\ & \qquad (y,t)\in\widetilde{G},\\ \widetilde{V}_0(y,t) &= \widetilde{\varphi}^2(y,t),\\ & \qquad (y,t)\in\widetilde{S}, \end{aligned}$$

where $\widetilde{\varphi}^{\,2}(y,t) = \varphi^2_y(y,t) = \varphi^2(X(y,t),t)$, $(y,t)\in\widetilde{S}$, $\widetilde{\varphi}^{\,2}(y,0) = \varphi^2(y,0)$.

The functions $V(x,t)$, $(x,t)\in\overline{G}$ and $\widetilde{V}_0(y,t)$, $(y,t)\in\overline{\widetilde{G}}$ satisfy the relations

$$\begin{aligned} \left|V(x,t) - \widetilde{V}_{0x}(x,t)\right| &\le M\delta^{1-\nu}, \quad \varepsilon\ge\delta; \qquad (3.16)\\ \left|V(x,t) - \widetilde{V}_{0x}(x,t)\right| &\le M\delta, \quad \varepsilon\le\delta, \quad (x,t)\in\overline{G};\\ \left|\frac{\partial^{k+k_0}}{\partial y^k\partial t^{k_0}}\widetilde{V}_0(y,t)\right| &\le M\varepsilon^{2k_0}\delta^{-k-2k_0}\left(1+\varepsilon\delta^{-1}t^{1/2}\right)^{-2k_0-k-1}\times\\ &\times\exp\left(-m_2\delta^{-2}\left(1+\varepsilon\delta^{-1}t^{1/2}\right)^{-2}y^2\right), \ \varepsilon\ge\delta;\\ \left|\frac{\partial^{k+k_0}}{\partial y^k\partial t^{k_0}}\widetilde{V}_0(y,t)\right| &\le M\delta^{-k}\exp\left(-m_1\delta^{-1}y\right), \quad \varepsilon\le\delta, \quad (y,t)\in\overline{\widetilde{G}}, \end{aligned}$$

where m_1 is an arbitrary number, m_2 is any number satisfying the condition $m_2 < m_2^0$, $m_2^0 = 4^{-1}\min\limits_{\overline{\widetilde{G}}}[\,A^{-1}(y,t)\widetilde{p}(y,t)\,]$, $\nu = \nu_{(3.11)}$.

THEOREM 2. *Let the conditions* (2.2) *hold. Then the solution of problem* (2.3) *satisfies the estimate* (3.14a). *If, besides this, the conditions* (3.5), (3.6) *are valid, then the functions* $U(x,t)$, $V(x,t)$, *that are the components of representation* (3.12), *satisfy the estimates* (3.14b) *and* (3.16).

REMARK. For small value of the parameters ε and/or δ, the function $\widetilde{V}_{0x}(x,t)$ is the main term in the asymptotic expansion of the singular part of the solution for boundary value problem (2.3). To compute the function $\widetilde{V}_{0x}(x,t)$, it is convenient to use a boundary value problem close to problem (3.15). From the variables x, t we pass to the variables η, t: $\eta = \eta(x,t)$, $\eta = Y(0,t) + (\partial/\partial x)Y(0,t)x$. By $\widehat{V}_0(\eta,t)$, $(\eta,t)\in\overline{\widehat{G}}$, where $\overline{\widehat{G}} = \overline{G}_\eta$, we denote the solution of the problem

$$\begin{aligned} L_{(3.17)}\widehat{V}_0(\eta,t) &\equiv \left\{\varepsilon^2\widehat{A}(t)\frac{\partial^2}{\partial\eta^2} - \widehat{c}(t) - \widehat{p}(t)\frac{\partial}{\partial t}\right\}\widehat{V}_0(\eta,t) = 0, \quad (\eta,t)\in\widehat{G},\\ & \qquad\qquad (3.17)\\ \widehat{V}_0(\eta,t) &= \widehat{\varphi}^2(\eta,t), \quad (\eta,t)\in\widehat{S}. \end{aligned}$$

Here $\widehat{G}^0 = G^0_\eta$ is the image of the set G^0, generally saying, $\widehat{G}^0 \neq \widetilde{G}^0$; $\widehat{A}(t) = A(0,t)$, $\widehat{\varphi}^2(\eta(x,t),t) = \varphi^2(x,t)$, $(x,t) \in S$, $\widehat{\varphi}^2(\eta,0) = \varphi^2(\eta,0)$, $\widehat{v}(t) = \widetilde{v}(0,t)$, where $\widehat{v}(t)$ and $\widetilde{v}(0,t)$ are respectively one of the functions $\widehat{c}(t),\ldots,\widetilde{p}(0,t)$. Thus, the coefficients of the equations and the initial data for problems (3.15) and (3.17) are coincident; however, in the case of problem (3.17) the relation between the original data x, t and the variables η, t is considerably simpler than that between x, t and y, t. This fact is used for the construction of special difference schemes. The function $\widehat{V}_{0x}(x,t) = \widehat{V}_0(\eta(x,t),t)$, $(x,t) \in \overline{G}$ satisfies estimates similar to (3.16).

4. Difference schemes for problem (2.1)

4.1. Consider the classical finite difference scheme for problem (2.1).
On the set $\overline{G}$ we introduce the mesh

$$\overline{G}_h = \overline{D}_h \times \overline{\omega}_0 = \overline{\omega} \times \overline{\omega}_0, \tag{4.1}$$

where $\overline{\omega}$, $\overline{\omega}_0$ are meshes in the intervals $[-d,d]$ and $[0,T]$, generally the meshes $\overline{\omega}$, $\overline{\omega}_0$ are nonuniform. Assume $h^i = x^{i+1} - x^i$, x^i, $x^{i+1} \in \overline{\omega}$, $h^k_t = h_t(t^k) \equiv t^{k+1} - t^k$, $t^k, t^{k+1} \in \overline{\omega}_0$, $h = \max_i h^i$, $h_t = \max_k h^k_t$. We denote by $N+1$ and N_0+1 the number of nodes in the meshes $\overline{\omega}$ and $\overline{\omega}_0$, respectively. Let $h \le MN^{-1}$, $h_t \le MN_0^{-1}$. We will also consider the meshes

$$\overline{G}_h = \overline{G}_{h(4.1)}, \tag{4.2}$$

where the mesh $\overline{\omega}_0$ is uniform.

With problem (2.1) we associate the difference scheme on the mesh $\overline{G}_h$:

$$\Lambda_{(4.3)} z(x,t) = f(x,t),\ (x,t) \in G_h, \quad z(x,t) = \varphi(x,t),\ (x,t) \in S_h. \tag{4.3}$$

Here $G_h = G \cap \overline{G}_h$, $S_h = S \cap \overline{G}_h$,

$$\begin{aligned}\Lambda_{(4.3)} z(x,t) \quad \equiv \quad & \{\varepsilon^2\{a(x,t)\delta_{\overline{x}\widehat{x}} + \left[b^+(x,t)\delta_x + b^-(x,t)\delta_{\overline{x}}\right]\} + \\ & + \left[b_2^{1+}(x,t)\delta_{x2} + b_2^{1-}(x,t)\delta_{\overline{x2}}\right] - c(x,t) - p(x,t)\delta_{\overline{t}}\} z(x,t),\end{aligned}$$

$v^+(x,t) = 2^{-1}(v(x,t) + |v(x,t)|)$, $v^-(x,t) = 2^{-1}(v(x,t) - |v(x,t)|)$, $\delta_x z(x,t)$, $\delta_{\overline{x}} z(x,t)$, $\delta_{\overline{t}} z(x,t)$, $\delta_{\overline{x}\widehat{x}} z(x,t)$ are the first and second difference derivatives on nonuniform meshes, for example,

$$\delta_{\overline{x}\widehat{x}} z(x,t) = 2\left(h^{i-1} + h^i\right)^{-1}\left(\delta_x z(x,t) - \delta_{\overline{x}} z(x,t)\right),\ x = x^i.$$

The maximum principle [6] is valid for difference scheme (4.3), (4.1). Using the maximum principle, we get (ε,δ)-uniform boundedness of the solution of problem (4.3), (4.1): $|z(x,t)| \le M$, $(x,t) \in \overline{G}_h$. The convergence of scheme (4.3), (4.1) can be investigated similarly to [4], [9], [10]. Taking into account estimate (3.1), we obtain

$$|u(x,t) - z(x,t)| \le M\left[\varepsilon^2\delta^{-3}N^{-1} + (1+\varepsilon^4\delta^{-4})N_0^{-1}\right], \quad (x,t) \in \overline{G}_h, \tag{4.4}$$

that is, difference scheme (4.3), (4.1) converges, uniformly with respect to the parameter ε, for fixed values of the parameter δ.

In the case of conditions (3.5), (3.6) we use estimates (3.9). For the approximate solution we get the estimate

$$
\begin{aligned}
|u(x,t) - z(x,t)| &\le M\left[\varepsilon^2\delta^{-3}N^{-1} + \left(1+\varepsilon^4\delta^{-4}\right)N_0^{-1}\right], \quad \varepsilon \ge \delta; \qquad (4.5)\\
|u(x,t) - z(x,t)| &\le M\left[\delta^{-1}N^{-1} + N_0^{-1}\right], \quad \varepsilon \le \delta,\ (x,t) \in \overline{G}_h.
\end{aligned}
$$

THEOREM 3. *Let the solution of boundary value problem* (2.1) *satisfy the estimates of Theorem* 1. *Then the solution of finite difference scheme* (4.3), (4.1) *converges ε-uniformly for the fixed values of the parameter δ. The discrete solution satisfies the estimates* (4.4) *and* (4.5).

For boundary value problem (2.1) we wish to construct the difference scheme which make it possible to find the approximations convergent (ε,δ)-uniformly.

4.2. To solve the problem for small values of the parameters ε and δ, we will use the method of the additive splitting of the singularities (cf. [5]). The solution of the problem is approximated by $u^0(x,t) = U(x,t) + V_0(x,t)$, $(x,t) \in \overline{G}$, where $U(x,t) = U_{(3.7)}(x,t)$, $V_0(x,t) = V_{0(3.10)}(x,t)$, $(x,t) \in \overline{G}$.

We approximate boundary value problem (3.8a) by the difference scheme

$$\Lambda_{(4.3)} z(x,t) = f(x,t),\ (x,t) \in G_h, \quad z(x,t) = \varphi^1(x,t),\ (x,t) \in S_h, \qquad (4.6)$$

where $\overline{G}_h = \overline{G}_{h(4.1)}$. To solve problem (3.10), we use the difference scheme

$$
\begin{aligned}
\Lambda_{(4.7)} z(x,t) \equiv \{&\varepsilon^2 a(0,t)\delta_{\overline{x}\widehat{x}} - c(0,t) - \qquad (4.7)\\
&-p(0,t)\delta_{\overline{t}}\} z(x,t) = 0, \quad (x,t) \in G_h,\\
z(x,t) &= \varphi^2(x,t), \quad (x,t) \in S_h.
\end{aligned}
$$

Here $\overline{G}_h = \overline{G}_{h(4.8)}$ is the mesh, generally different from the mesh in problem (4.6)

$$\overline{G}_h = \overline{\omega} \times \overline{\omega}_0, \qquad (4.8)$$

where $\overline{\omega}_0 = \overline{\omega}_{0(4.1)}$. We denote by $\overline{z}(x,t)$, $(x,t) \in \overline{G}_{2h}$, where $\overline{G}_{2h} = [-d,d] \times [0,T]$, the interpolant linear with respect to each of the variables and constructed from the values of the solution of problem (4.6), (4.1). The approximate solution of problem (2.1) is defined by the relation

$$z(x,t) = \overline{z}_{(4.6)}(x,t) + z_{(4.7)}(x,t), \quad (x,t) \in \overline{G}_{h(4.8)}. \qquad (4.9)$$

The function $z_{(4.9)}(x,t)$, $(x,t) \in \overline{G}_{h(4.8)}$ is called the solution of difference scheme (4.6), (4.1), (4.7), (4.8). For solving problem (3.10) we use the meshes condensing in the neighbourhood of the set S^*.

4.3. Assume

$$\varepsilon \le \delta, \quad \varepsilon \in (0,1], \quad \delta \in (0,d]. \qquad (4.10)$$

On the set $\overline{G}$ we introduce the special mesh

$$\overline{G}_h = \overline{D}_h^{\,c} \times \overline{\omega}_0 = \overline{\omega}^{\,c} \times \overline{\omega}_0, \tag{4.11a}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(4.2)}$, $\overline{\omega}^{\,c} = \overline{\omega}^{\,c}(\sigma)$ is a piecewise-uniform mesh on $[-d, d]$, σ is a parameter depending on δ and N. The step of the mesh $\overline{\omega}^{\,c}$ is constant on the intervals $[-d, -\sigma]$, $[\sigma, d]$ and $[-\sigma, \sigma]$, and equal to $h^{(2)} = 4(d-\sigma)N^{-1}$ and $h^{(1)} = 4\sigma N^{-1}$, respectively. The value σ is chosen to satisfy the relation

$$\sigma = \sigma_{(4.11)}(\delta, N) = \min\left[2^{-1}d,\ m^{-1}\delta \ln N\right], \tag{4.11b}$$

where $m = m_{(4.11)}$ is an arbitrary number.

Taking into account the estimates of Theorem 1 for difference scheme (4.6), (4.2), (4.7), (4.1), we find the estimate

$$|u(x,t) - z(x,t)| \le M\left[N^{-1}\ln N + N_0^{-1} + \delta\right],\ (x,t) \in \overline{G}_{(4.11)},\ \varepsilon \le \delta. \tag{4.12}$$

Thus, under condition (4.10), the solution of the difference scheme converges to the solution of boundary value problem (2.1) for $\delta = o(1)$ (as $N, N_0 \to \infty$).

4.4. Let the condition

$$\varepsilon \ge \delta,\ \ \varepsilon \in (0,1],\ \ \delta \in (0,d]. \tag{4.13}$$

be valid. On the set $\overline{G}$ we introduce the mesh which is condensed in the neighbourhood of the set S^* and also for small values of t:

$$\overline{G}_h = \overline{D}_h^{\,c} \times \overline{\omega}_0^c = \overline{\omega}^{\,c} \times \overline{\omega}_0^c, \tag{4.14a}$$

where $\overline{\omega}^{\,c} = \overline{\omega}^{\,c}(\sigma)$, $\overline{\omega}_0^c = \overline{\omega}_0^c(\sigma_0)$ are piecewise uniform meshes in the intervals $[-d, d]$ and $[0, T]$ respectively, σ_0, σ_1 are parameters depending on ε, δ, N_0, N. Assume $\sigma_1 = \sigma_{(4.11)}(\delta, N)$, $\overline{\omega}^{\,c}(\sigma_1) = \overline{\omega}^{\,c}_{(4.11)}(\sigma_1)$. We define the value σ_0 by the relation

$$\sigma_0 = \sigma_0(\varepsilon, \delta, N, N_0) = \min\left[\,2^{-1}T,\, \varepsilon^{-2}\delta^2 N^{2/3},\, \varepsilon^{-2}\delta^2 N_0^{2/5}\,\right]. \tag{4.14b}$$

The step of the mesh $\overline{\omega}_0^c$ is constant on the intervals $[0, \sigma_0]$ and $[\sigma_0, T]$, and equal to $h_0^{(1)} = 2\sigma_0 N_0^{-1}$ and $h_0^{(2)} = 2(T - \sigma_0)N_0^{-1}$, respectively. Thus, the mesh $\overline{G}_{h(4.14)}$ has been constructed.

Taking into account the *a-priori* estimates of the boundary value problem, for the solution of difference scheme (4.6), (4.2), (4.7), (4.14) we obtain the estimate

$$|u(x,t) - z(x,t)| \le M\left[N^{-1/3} + N_0^{-1/5} + \delta^{1-\nu}\right], \tag{4.15}$$

$$(x,t) \in \overline{G}_{h(4.14)},\ \ \varepsilon \ge \delta,\ \ \nu = \nu_{(3.11)}.$$

Thus, in case of condition (4.13), difference scheme (4.6), (4.2), (4.7), (4.14) converges, as $N, N_0 \to \infty$, to the solution of the boundary value problem for $\delta = o(1)$.

4.5. The estimates (4.5), (4.12), (4.15) imply that difference schemes (4.3), (4.2); (4.6), (4.2), (4.7), (4.11) and (4.6), (4.2), (4.7), (4.14) make it possible to construct the approximate solution convergent to the solution of the boundary value problem (ε, δ)-uniformly. We define the function $z_{(4.16)}(x,t)$, $(x,t) \in \overline{G}_{h(4.16)}$, assuming

$$z(x,t) = z_{(4.3,4.2)}(x,t), \quad (x,t) \in \overline{G}_{h(4.16)}, \quad \overline{G}_{h(4.16)} = \overline{G}_{h(4.2)} \tag{4.16}$$

$$\text{for either} \quad \varepsilon \le \delta,\ \delta \ge \Psi_1(N, N_0), \ \text{ or } \ \varepsilon > \delta,\ \delta \ge \Psi_2(N, N_0);$$

$$z(x,t) = z_{(4.6,4.2,4.7,4.11)}(x,t), \quad (x,t) \in \overline{G}_{h(4.16)}, \quad \overline{G}_{h(4.16)} = \overline{G}_{h(4.11)}$$

$$\text{for} \ \ \varepsilon \le \delta,\ \delta < \Psi_1(N, N_0);$$

$$z(x,t) = z_{(4.6,4.2,4.7,4.14)}(x,t), \quad (x,t) \in \overline{G}_{h(4.16)}, \quad \overline{G}_{h(4.16)} = \overline{G}_{h(4.14)}$$

$$\text{for} \ \ \varepsilon > \delta,\ \delta < \Psi_2(N, N_0),$$

where $\Psi_1(N, N_0) = N^{-1/2}$, $\Psi_2(N, N_0) = (N^{-1} + N_0^{-1})^{1/5}$. We call the function $z_{(4.16)}(x,t)$ the solution of difference scheme (4.3), (4.6), (4.2), (4.7), (4.11), (4.14).

The solution of this scheme converges (ε, δ)-uniformly with error bounds

$$\left|u(x,t) - z_{(4.16)}(x,t)\right| \le M\left(N^{-1/2} + N_0^{-1}\right),\ (x,t) \in \overline{G}_h,\ \varepsilon \le \delta; \tag{4.17}$$

$$\left|u(x,t) - z_{(4.16)}(x,t)\right| \le M\left(N^{-1} + N_0^{-1}\right)^{(1-\nu)/5},\ (x,t) \in \overline{G}_h,\ \varepsilon \ge \delta,\ \nu = \nu_{(3.11)}.$$

THEOREM 4. *Let the solution of boundary value problem* (2.1) *satisfy the estimates of Theorem* 1*. Then the solution of finite difference scheme* (4.3), (4.6), (4.2), (4.7), (4.11), (4.14) *converges to the solution of the boundary value problem (ε, δ)-uniformly. The discrete solution satisfies estimates* (4.17)*.*

REMARK. For solving the discrete problems (4.3) and (4.6) one can use, instead of the mesh $\overline{G}_{h(4.2)}$, the meshes $\overline{G}_{h(4.11)}$ for $\varepsilon \le \delta$ and $\overline{G}_{h(4.14)}$ for $\varepsilon > \delta$. The statement of the theorem holds also in this case.

5. DIFFERENCE SCHEMES FOR BOUNDARY VALUE PROBLEM (2.3)

For problem (2.3) we will construct the classical and special difference schemes and investigate their convergence.

5.1. We approximate the boundary value problem (2.3) by the discrete problem

$$\Lambda_{(5.1)} z(x,t) = f(x,t),\ (x,t) \in G_h, \quad z(x,t) = \varphi(x,t),\ (x,t) \in S_h. \tag{5.1}$$

Here $\overline{G}_h = \overline{G}_{h(4.1)}$,

$$\begin{aligned} \Lambda_{(5.1)} z(x,t) \ \equiv \ & \{\varepsilon^2\{a(x,t)\delta_{\overline{x}\widehat{x}} + \left[b^+(x,t)\delta_x + b^-(x,t)\delta_{\overline{x}}\right]\} + \\ & + \left[b^{1+}(x,t)\delta_x + b^{1-}(x,t)\delta_{\overline{x}}\right] - c(x,t) - p(x,t)\delta_{\overline{t}}\} z(x,t). \end{aligned}$$

Because of the maximum principle, the solution of difference scheme (5.1), (4.1) is bounded (ε,δ)-uniformly. Taking account the estimate of Theorem 2, we find

$$
\begin{aligned}
|u(x,t)-z(x,t)| &\le M\left[\varepsilon^8\delta^{-9}N^{-1}+\varepsilon^4\delta^{-6}N_0^{-1}\right], \quad \varepsilon\ge\delta; \\
|u(x,t)-z(x,t)| &\le M\left[\delta^{-1}N^{-1}+\delta^{-2}N_0^{-1}\right], \quad \varepsilon\le\delta,\ (x,t)\in\overline{G}_h.
\end{aligned}
\tag{5.2}
$$

5.2. To construct the special difference scheme, we make an additive splitting of the singularities. The solution of the problem is approximated by the function $u^0(x,t)=U(x,t)+V_0(x,t)$, $(x,t)\in\overline{G}$. Here $U(x,t)=U_{(3.12)}(x,t)$, $V_0(x,t)=\widehat{V}_{0(3.17)x}(x,t)$, $(x,t)\in\overline{G}$.

We approximate the boundary value problem (3.13a) on the mesh $\overline{G}_{h(4.2)}$ by the difference scheme

$$
\Lambda_{(5.1)}z(x,t)=f(x,t),\ (x,t)\in G_h,\quad z(x,t)=\varphi^1(x,t),\ (x,t)\in S_h, \tag{5.3}
$$

Let us construct the difference scheme for the numerical solution of problem (3.17). On the set $\overline{\widehat{G}}$ (with the "curvilinear" boundary $\widehat{S}$) we introduce the mesh

$$
\overline{\widehat{G}}_h=\widehat{G}_h\cup\widehat{S}_h,\quad \widehat{G}_h=\widehat{G}\cap\{\omega\times\overline{\omega}_0\}, \tag{5.4}
$$

where $\overline{\omega}_0=\overline{\omega}_{0(4.1)}$, ω is a mesh on the axis η, generally speaking, nonuniform. We denote by $N+1$ the number of nodes in the mesh ω on the minimum interval of the axis η onto which the set $\overline{\widehat{G}}$ is projected. The set $\widehat{S}_h=\widehat{S}_{0h}\cup\widehat{S}_{1h}$; the set $\widehat{S}_{1h}$ are formed by the points of intersection of the surface $\widehat{S}$ with the straight lines passing through the nodes of the set $\widehat{G}_h$ in a parallel way with the axis η; the set $\widehat{S}_{0h}$ is the low base defined by the relation: $\widehat{S}_{0h}=\overline{\widehat{G}}\cap\{\,\omega\times[\,t=0\,]\,\}$.

On the mesh $\overline{\widehat{G}}_h$, for problem (3.17) we introduce the difference scheme

$$
\Lambda_{(5.5)}\widehat{z}(\eta,t)=0,\ (\eta,t)\in\widehat{G}_h,\quad \widehat{z}(\eta,t)=\widehat{\varphi}^2(\eta,t),\ (\eta,t)\in\widehat{S}_h, \tag{5.5}
$$

where $\Lambda_{(5.5)}\widehat{z}(\eta,t)\equiv\{\varepsilon^2\widehat{A}(t)\delta_{\overline{\eta}\widehat{\eta}}-\widehat{c}(t)-\widehat{p}(t)\delta_{\overline{t}}\}\widehat{z}(\eta,t)$.

We define the approximate solution of problem (2.3) by the relation

$$
z(x,t)=\overline{z}_{(5.3,4.2)}(x,t)+\widehat{z}_{(5.5,5.4)x}(x,t),\quad (x,t)\in\overline{\widehat{G}}_{hx}. \tag{5.6}
$$

Here $\overline{\widehat{G}}_{hx}$ is an image of the set $\overline{\widehat{G}}_h$ under the mapping $x=x(\eta,t)$, $(\eta,t)\in\overline{\widehat{G}}$, which is inverse to the mapping $\eta=\eta(x,t)$; $z_{(5.3,4.2)}(x,t)$, $(x,t)\in\overline{G}_h$ and $\widehat{z}_{(5.5,5.4)}(\eta,t)$, $(\eta,t)\in\overline{\widehat{G}}_h$ are the solutions of problems (5.3), (4.2 and (5.5), (5.4), respectively, $\overline{z}_{(5.3,4.2)}(x,t)$ is the bilinear interpolant with respect to x and t, and $\widehat{z}_{(5.5,5.4)x}(x,t)=\widehat{z}_{(5.5,5.4)}(\eta(x,t),t)$. We shall call the function $z_{(5.6)}(x,t)$, $(x,t)\in\overline{\widehat{G}}_{hx}$, the solution of difference scheme (5.3), (4.2), (5.5), (5.4).

To solve problem (3.17), we use the meshes condensing in a neighbourhood of the set $\widehat{S}^*$, where $\widehat{S}^*=\{(\eta,t):\ \eta=0,\ t\in(0,T]\}$.

5.3. Let the condition $\varepsilon \le \delta$, $\varepsilon \in (0,1]$, $\delta \in (0,d]$ be valid. On the set $\overline{\widehat{G}}$ we introduce the special mesh

$$\overline{\widehat{G}}_h = \overline{\widehat{G}}_{h(5.4)}, \tag{5.7}$$

where $\omega = \omega^c(\sigma)$ is a piecewise uniform mesh, similar to the mesh $\overline{\omega}^c_{(4.11)}$, condensing in a neighbourhood of the points $\eta = 0$. The distribution of nodes of the meshes $\omega^c_{(5.7)}(\sigma)$ in the interval $[-\sigma,\sigma]$ and outside this interval is assumed to be uniform. The stepsizes $h^{(1)}_{(5.7)}$ and $h^{(2)}_{(5.7)}$, respectively in the interval $[-\sigma,\sigma]$ and outside of it, are defined by the relations: $h^{(1)}_{(5.7)} = 4\sigma N^{-1}$, $h^{(2)}_{(5.7)} = 2(d_0 - 2\sigma)N^{-1}$, where $\sigma = \sigma_{(4.11)}(\delta, N_1)$, d_0 is the length of that interval on the axis η onto which the set $\overline{\widehat{G}}$ is projected.

For the solution of difference scheme (5.3), (4.2), (5.5), (5.7) we have the estimate

$$|u(x,t) - z(x,t)| \le M\left[N^{-1}\ln N + N_0^{-1} + \delta\right],\ (x,t) \in \overline{\widehat{G}}_{hx},\ \overline{\widehat{G}}_h = \overline{\widehat{G}}_{h(5.7)},\ \varepsilon \le \delta.$$

If the condition $\varepsilon \ge \delta$, $\varepsilon \in (0,1]$, $\delta \in (0,d]$ holds, then we construct the mesh on $\overline{\widehat{G}}$ which is condensing in a neighbourhood of the set $\widehat{S}^*$ and also for small values of t:

$$\overline{\widehat{G}}_h = \overline{G}_{h(4.1)} \ \ \{ \text{ for } \ (x,t) \equiv (\eta,t)\ \}, \tag{5.8}$$

where $\omega = \omega^c_{(5.7)}(\sigma_1)$ with $\sigma_1 = \sigma_{(4.11)}(\delta, N)$, $\overline{\omega}_0 = \overline{\omega}^c_{0(4.14)}$.

For the solution of difference scheme (5.3), (4.2), (5.5), (5.8) we have the estimate

$$|u(x,t) - z(x,t)| \le M\left[N^{-1/3} + N_0^{-1/5} + \delta^{1-\nu}\right],\ \ (x,t) \in \overline{\widehat{G}}_{hx},$$
$$\overline{\widehat{G}}_h = \overline{\widehat{G}}_{h(5.8)},\ \ \varepsilon \ge \delta,\ \ \nu = \nu_{(3.11)}.$$

5.4. We will construct the (ε,δ)-uniform approximation for the solution of boundary value problem (2.3). We define the function $z_{((5.9))}(x,t)$, $(x,t) \in \overline{G}_{h((5.9))}$, assuming

$$z(x,t) = z_{(5.1,4.2)}(x,t), \quad (x,t) \in \overline{G}_{h(5.9)},\ \overline{G}_{h(5.9)} = \overline{G}_{h(4.2)}. \tag{5.9}$$
$$\text{for either } \ \varepsilon \le \delta,\ \delta \ge \Psi_3(N,N_0), \text{ or } \ \varepsilon > \delta,\ \delta \ge \Psi_4(N,N_0);$$
$$z(x,t) = z_{(5.3,4.2,5.5,5.7)}(x,t), \quad (x,t) \in \overline{G}_{h(5.9)},\ \overline{G}_{h(5.9)} = \overline{\widehat{G}}_{h(5.7)x}$$
$$\text{for } \ \varepsilon \le \delta,\ \delta < \Psi_3(N,N_0);$$

$$z(x,t) = z_{(5.3,4.2,5.5,5.8)}(x,t), \quad (x,t) \in \overline{G}_{h(5.9)},\ \overline{G}_{h(5.9)} = \overline{\widehat{G}}_{h(5.8)x}$$
$$\text{for } \ \varepsilon > \delta, \delta < \Psi_4(N,N_0);$$

where $\Psi_3(N,N_0) = (N^{-1} + N_0^{-1})^{1/3}$, $\Psi_4(N,N_0) = (N^{-1} + N_0^{-1})^{1/10}$. We shall call the function $z_{(5.9)}(x,t)$, $(x,t) \in \overline{G}_{h(5.9)}$, the solution of scheme (5.1), (5.3), (4.2), (5.5), (5.7), (5.8).

The solution of this scheme converges (ε, δ)-uniformly:

$$\begin{aligned} \left|u(x,t) - z_{(5.9)}(x,t)\right| &\leq M\left(N^{-1} + N_0^{-1}\right)^{1/3}, \quad (x,t) \in \overline{G}_h,\ \varepsilon \leq \delta; \\ \left|u(x,t) - z_{(5.9)}(x,t)\right| &\leq M\left(N^{-1} + N_0^{-1}\right)^{(1-\nu)/10}, \quad (x,t) \in \overline{G}_h,\ \varepsilon \geq \delta. \end{aligned} \tag{5.10}$$

THEOREM 5. *Let the solution of boundary value problem* (2.3) *satisfy the estimates of Theorem* 2. *Then the solution of finite difference scheme* (5.1), (5.3), (4.2), (5.5), (5.7), (5.8) *converges to the solution of the boundary value problem (ε, δ)-uniformly. The solution of the difference scheme satisfies estimates* (5.10).

5.5. We assume that, as for difference scheme (5.1), (5.3), (4.2), (5.5), (5.7), (5.8), the coefficients of the equation for problem (3.17) and the functions $s_0(t) = Y(0,t)$, $s_1(t) = (\partial/\partial x)Y(0,t)$, $t \in \overline{H}$ (where $H = (0,T]$), which define the transformation $\eta = \eta(x,t)$, can be found in the explicit form. Therefore, this difference scheme is not constructive. We give the difference scheme in which the coefficients of the difference equations and the functions $s_0(t)$, $s_1(t)$ are being found numerically. We call the curve $x = s_0(t)$, $t \in \overline{H}$, the trace of the local disturbance in the initial conditions (or, shortly, by the disturbance trace). Let us construct the discrete approximation for the function $s_0(t)$, $t \in H$.

Note that the coefficients of equation (3.17) are determined by the relations

$$\widehat{A}(t) = A_{((3.4))}(0,t) = a(s_0(t),t)[s_1(t)]^2, \ \widehat{v}(t) = v(s_0(t),t),$$

where $v(x,t)$ and $\widehat{v}(t)$ are the functions $c(x,t)$, $p(x,t)$ and $\widehat{c}(t)$, $\widehat{p}(t)$, respectively; $s_0(t) = Y(0,t)$, $s_1(t) = (\partial/\partial x)Y(0,t)$, $t \in \overline{H}$.

We approximate the functions $s_i(t)$, $i = 0,1$ in the following way. The function $s_0(t)$, $t \in \overline{H}$ is approximated by the function $q_0(t)$, $t \in \overline{H}_{2h}$. Assume that we know the function $q_0(t)$, $t \in \overline{H}_{2h}$ for $t \leq t^0$, $t^0 \in \overline{\omega}_0$, $\overline{\omega}_0 = \overline{\omega}_{0(4.1)}$, $t^0 < T$, and also $q_0(0) = 0$. We construct $q_0(t)$ for $t \leq \widehat{t}^0$, $\widehat{t}^0 = t^0 + h_t(t^0)$. By $q(t)$, $t \in [t^0, \widehat{t}^0]$, we denote the solution of the problem

$$\frac{d}{dt}q(t) = -\left[p(q(t),t)\right]^{-1} b^1(q(t),t), \quad t \in (t^0, \widehat{t}^0], \tag{5.11a}$$

$$q(t^0) = q(t^0 - 0) \quad \text{for} \quad t^0 > 0,$$

$$q(t^0) = 0 \qquad \text{for} \quad t^0 = 0.$$

Thus, $x = q(t)$, $t \in [0, \widehat{t}^0]$ is the characteristic passing through the point $(0,0)$. The function $q_0(t)$ for $t \in (t^0, \widehat{t}^0]$, $t^0, \widehat{t}^0 \in \overline{\omega}_0$, is defined by the relation

$$q_0(t) = q(t), \quad t \in (t^0, \widehat{t}^0], \quad t^0, \widehat{t}^0 \in \overline{\omega}_0. \tag{5.11b}$$

Continuing this process, we find the function $q_0(t)$, $t \in \overline{H}$. The function $q_0(t)$ converges to the function $s_0(t)$: $|s_0(t) - q_0(t)| \le M[N^{-1} + N_0^{-1}]$, $t \in \overline{H}$.

We approximate the function $s_1(t)$, $t \in \overline{H}$ by the function $Q_1(t)$, $t \in \overline{H}$. Let the function $Q_1(t)$, $t \in \overline{H}$ be known for $t \le t^0$, $t^0 \in \overline{\omega}_0$, and also $Q_1(0) = 1$. Construct the function $Q_1(t)$ for $t \in (t^0, \hat{t}^0]$. We denote by $Q(t)$, $t \in [t^0, \hat{t}^0]$, the solution of the problem

$$LQ(t) \equiv -p(q(t),t)\frac{d}{dt}Q(t) = F(t, q(t), Q(t)), \quad t \in (t^0, \hat{t}^0], \tag{5.12a}$$

$$Q(t^0) = Q_1(t^0 - 0) \ \text{ for } \ t^0 > 0,$$

$$Q(t^0) = Q_1(t^0) \qquad \text{for } \ t^0 = 0,$$

where
$$F(t, q(t), Q(t)) = -\frac{\partial}{\partial x} b^1(q(t),t)Q(t) + \frac{\partial}{\partial x} p(q(t),t)Q^0(t),$$

$$Q^0(t) = [p(q(t),t)]^{-1}\, b^1(q(t),t)Q(t), \quad q(t) = q_{(5.11)}(t).$$

The function $Q(t)$, $t \in [t^0, \hat{t}^0]$ approximates the derivative of the function $Y(x,t)$, in x, on the characteristic $x = q_{(5.11)}(t)$, $t \in [t^0, \hat{t}^0]$. We define the function $Q_1(t)$, $t \in (t^0, \hat{t}^0]$ by

$$Q_1(t) = Q(t), \quad t \in (t^0, \hat{t}^0], \quad t^0,\, \hat{t}^0 \in \overline{\omega}_0. \tag{5.12b}$$

Continuing the process, we find the function $Q_1(t)$, $t \in \overline{H}$. The function $Q_1(t)$ converges to the function $s_1(t)$: $|s_1(t) - Q_1(t)| \le M[N^{-1} + N_0^{-1}]$, $t \in \overline{H}$.

Let us construct the mesh approximations of problems (5.11) and (5.12). We introduce the mesh on $\overline{H}$:

$$\overline{H}_h = \overline{\omega}_0, \tag{5.13}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(4.1)}$. With problems (5.11), (5.12) we associate the difference scheme (on the basis of explicit difference approximations)

$$\delta_t q^h(t) = -\left[p(q^h(t),t)\right]^{-1} b^1(q^h(t),t), \quad t \in \omega_0, \tag{5.14}$$

$$q^h(0) = 0, \quad s_0^h(\hat{t}) = q^h(\hat{t}), \quad t \in \overline{H}_h;$$

$$-p(q^h(t),t)\delta_t Q^h(t) = F_{(5.12)}(t, q^h(t), Q^h(t)), \quad t \in \omega_0,$$

$$Q^h(0) = 1, \quad s_1^h(\hat{t}) = Q^h(\hat{t}), \quad t \in \overline{H}_h, \quad t, \hat{t} \in \overline{\omega}_0.$$

We shall call the functions $s_i^h(t)$, $t \in \overline{H}_h$, $i = 0, 1$, the solution of difference scheme (5.14), (5.13).

Assume the step $\overline{\omega}_0$ to be sufficiently small. The coefficients of each of the difference equations in (5.14), being written for the two-point stencil (see, e.g., [8]), have opposite signs and are commensurable in modulus.

The solution of discrete problem (5.14), (5.13) satisfies the estimate

$$\left|s_i(t) - s_i^h(t)\right| \le M[N^{-1} + N_0^{-1}], \quad t \in \overline{H}_h, \quad i = 0, 1. \tag{5.15}$$

5.6. Now we construct the difference scheme for the solution of problem (2.3). The mapping $\eta = \eta(x,t)$, $(x,t) \in \overline{G}$ corresponds here to the mapping $\xi = \xi(x,t)$, $\xi = s_0^h(t) + s_1^h(t)x$, $t \in \overline{H}_h$. On the set $\overline{G}$ we construct the mesh related to this last mapping. Denote by $Q = R \times \overline{H}_h$ the range of the variables ξ, t, and let ω^0 be a mesh on the axis ξ. Assume $\overline{Q}^h = \omega^0 \times \overline{H}_h$; $\overline{Q}^h_x = \{(x,t) : \xi = \xi(x,t),\ (\xi,t) \in \overline{Q}^h\}$. On the set $\overline{G}$ we define the mesh

$$\overline{G}^h = (\overline{G}^h_\xi)_x. \tag{5.16a}$$

The mesh

$$\overline{G}^h_\xi = G^h_\xi \cup S^h_\xi, \quad G^h_\xi = \widehat{G} \cap H_h, \tag{5.16b}$$

in the variables ξ, t corresponds to the mesh $\overline{G}^h$. The set S^h_ξ includes the set $S_{0\xi} \cap \overline{Q}^h$ and also the set of the points of intersection of the lines, which are parallel to the axis ξ and passing through the nodes of G^h_ξ, with the surface S_ξ.

The coefficients $\widehat{A}(t)$, $\widehat{c}(t)$, $\widehat{p}(t)$ in equation (3.17) correspond to the mesh functions

$$\widehat{A}^h(t) = a\left(s_0^h(t),t\right)[s_1^h(t)]^2, \quad \widehat{c}^h(t) = c\left(s_0^h(t),t\right), \quad \widehat{p}^h(t) = p(s_0^h(t),t), \quad t \in \overline{H}_h.$$

On the mesh $\overline{G}^h_{\xi(5.16)}$, for the solution of problem (3.17) we use the difference scheme

$$\Lambda_{(5.17)} z(\xi,t) = 0,\ (\xi,t) \in G^h_\xi, \quad z(\xi,t) = \varphi^2_\xi(\xi,t),\ (\xi,t) \in S^h_\xi, \tag{5.17}$$

where $\varphi^2_\xi(\xi,t) = \varphi^2(x(\xi,t),t)$, $\Lambda_{(5.17)} z(\xi,t) \equiv \{\varepsilon^2 \widehat{A}^h(t)\delta_{\overline{\xi}\widehat{\xi}} - \widehat{c}^h(t) - \widehat{p}^h(t)\delta_{\overline{t}}\} z(\xi,t)$.

We define the approximate solution of problem (2.3) by the relation

$$z(x,t) = \overline{z}_{(5.3,4.2)}(x,t) + z_{(5.17,5.16)x}(x,t),\ (x,t) \in \overline{G}^{\,h}, \tag{5.18}$$

where $\overline{G}^{\,h} = \overline{G}^{\,h}_{(5.16)}$, $z_{(5.17,5.16)x}(x,t) = z_{(5.17,5.16)}(\xi(x,t),t)$, $(x,t) \in \overline{G}^h_{(5.16)}$. We shall call the function $z_{(5.18)}(x,t)$, $(x,t) \in \overline{G}^h$, the solution of scheme (5.3), (4.2), (5.17), (5.16).

In case $\varepsilon \leq \delta$ we use the mesh

$$\overline{G}^h_\xi = \overline{G}^h_{\xi(5.16)}, \tag{5.19}$$

where $\overline{\omega}_0 = \overline{\omega}_{0(4.2)}$, $\overline{\omega}^0 = \overline{\omega}^{0c}(\sigma)$ is a piecewise uniform mesh condensing in a neighbourhood of the point $\xi = 0$; $\overline{\omega}^{0c}_{(5.19)}(\sigma) = \overline{\omega}^c_{(4.11)}(\sigma)$ for $\sigma = \sigma_{(4.11)}(\delta, N)$, $h^{(1)}_{(5.19)} = h^{(1)}_{(4.11)} = 4\sigma N^{-1}$, $h^{(2)}_{(5.19)} = 2(d_\xi - 2\sigma)N^{-1}$, d_ξ is the length of that interval on the axis ξ onto which the set $\overline{G}^h_\xi$ is projected. If $\varepsilon \geq \delta$, we use the mesh

$$\overline{G}^h_\xi = \overline{G}^h_{\xi(5.16)}, \tag{5.20}$$

where $\omega^0 = \omega^{0c}_{(5.19)}(\sigma_1)$ for $\sigma_1 = \sigma_{(4.11)}(\delta, N)$, $\quad \overline{\omega}_0 = \overline{\omega}^c_{0(4.14)}$.

For the solutions of problems (5.3), (4.2), (5.17), (5.19) and (5.3), (4.2), (5.17), (5.20) we obtain the estimates

$$\left|u(x,t) - z_{(5.3,4.2,5.17,5.19)}(x + s_0(t) - s_0^h(t), t)\right| \le M[N^{-1}\ln N + N_0^{-1} + \delta],$$

$$(x,t) \in \overline{G}^h_{(5.19)}, \quad \varepsilon \le \delta,$$

$$\left|u(x,t) - z_{(5.3,4.2,5.17,5.20)}(x + s_0(t) - s_0^h(t), t)\right| \le M[N^{-1/3} + N_0^{-1/5} + \delta^{1-\nu}],$$

$$(x,t) \in \overline{G}^h_{(5.20)}, \quad \varepsilon \ge \delta, \quad |x - s_0(t)| \le m;$$

$$\left|u(x,t) - z_{(5.3,4.2,5.17,5.19)}(x,t)\right| \le M[N^{-1}\ln N + N_0^{-1} + \delta],\ (x,t) \in \overline{G}^h_{(5.19)},\ \varepsilon \le \delta,$$

$$\left|u(x,t) - z_{(5.3,4.2,5.17,5.20)}(x,t)\right| \le M\left[N^{-1/3} + N_0^{-1/5} + \delta^{1-\nu}\right],$$

$$(x,t) \in \overline{G}^h_{(5.20)}, \quad \varepsilon \ge \delta, \quad |x - s_0(t)| \ge m, \quad \nu = \nu_{(3.11)},$$

where $\overline{G}^h_{(5.19)} = (\overline{G}^h_{\xi(5.19)})_x$, $\overline{G}^h_{(5.20)} = (\overline{G}^h_{\xi(5.20)})_x$. Thus, if the values of the parameters are small, the solutions of the difference schemes approximate (in the discrete maximum norm) the solution of problem (2.3) outside an m-neighbourhood of the trace; inside this m-neighbourhood of the trace, the discrete solutions approximate the exact ones at the points equidistant from the numerical and exact traces, respectively.

We will construct the (ε,δ)-uniform approximation for the solution of boundary value problem (2.3) ly. Define the function $z_{(5.21)}(x,t)$, $(x,t) \in \overline{G}_{h(5.21)}$, assuming

$$z(x,t) = z_{(5.1,4.2)}(x,t), \quad (x,t) \in \overline{G}_{h(5.21)},\ \overline{G}_{h(5.21)} = \overline{G}_{h(4.2)} \tag{5.21}$$

$$\text{for either } \ \varepsilon \le \delta,\ \delta \ge \Psi_3(N,N_0), \ \text{ or } \ \varepsilon > \delta,\ \delta \ge \Psi_4(N,N_0);$$

$$z(x,t) = z_{(5.3,4.2,5.17,5.19)}(x,t), \quad (x,t) \in \overline{G}_{h(5.21)},\ \overline{G}_{h(5.21)} = \overline{G}^h_{(5.19)}$$

$$\text{for } \ \varepsilon \le \delta,\ \delta < \Psi_3(N,N_0);$$

$$z(x,t) = z_{(5.3,4.2,5.17,5.20)}(x,t), \quad (x,t) \in \overline{G}_{h(5.21)},\ \overline{G}_{h(5.21)} = \overline{G}^h_{(5.20)}$$

$$\text{for } \ \varepsilon > \delta,\ \varepsilon < \Psi_4(N,N_0);$$

$\Psi_i(N,N_0) = \Psi_{i(5.9)}(N,N_0)$. We call the pair of the functions $z_{(5.21)}(x,t)$, $(x,t) \in \overline{G}_{h(5.21)}$, and $s^h_{0(5.14)}(t)$, $t \in \overline{H}_{h(5.13)}$, the solution of difference scheme (5.1), (5.3), (4.2), (5.17), (5.19), (5.20).

The solution of this scheme satisfies the estimates

$$\left|u(x,t) - z_{(5.21)}(x + s_0(t) - s_0^h(t), t)\right| \le M(N^{-1} + N_0^{-1})^{1/3}, \tag{5.22}$$

$$(x,t) \in \overline{G}_h, \quad \varepsilon \le \delta,$$

$$\left|u(x,t) - z_{(5.21)}(x + s_0(t) - s_0^h(t), t)\right| \le M(N^{-1} + N_0^{-1})^{(1-\nu)/10},$$
$$(x,t) \in \overline{G}_h, \quad \varepsilon \ge \delta, \quad \left|x - s_0^h(t)\right| \le m;$$
$$\left|u(x,t) - z_{(5.21)}(x,t)\right| \le M(N^{-1} + N_0^{-1})^{1/3}, \ (x,t) \in \overline{G}_h, \ \varepsilon \le \delta,$$
$$\left|u(x,t) - z_{(5.21)}(x,t)\right| \le M(N^{-1} + N_0^{-1})^{(1-\nu)/10},$$
$$(x,t) \in \overline{G}_h, \quad \varepsilon \ge \delta, \quad \left|x - s_0^h(t)\right| \ge m,$$

for the function $s_0^h(t)$ the estimate (5.15) holds. To compute the function $s_i^h(t)$, $t \in \overline{H}_{h(5.13)}$, $i = 0, 1$ by the scheme on the mesh $\overline{G}^h_{(5.19)}$ (on the mesh $\overline{G}^h_{(5.20)}$), we use the mesh $\overline{H}_{h(5.13)}$, where $\overline{\omega}_0 = \overline{\omega}_{0(5.19)}$ (where $\overline{\omega}_0 = \overline{\omega}_{0(5.20)}$).

Thus, as N, $N_0 \to \infty$, the function $z_{(5.21)}(x,t)$ converges (ε, δ)-uniformly to the solution of boundary value problem (2.3) outside the m-neighbourhood of the curve $x = s_0(t)$, $t \in \overline{H}$; whereas in the neighbourhood of this curve the values of the function $z_{(5.21)}(x,t)$ at the points $(x,t) \in \overline{G}_{h(5.21)}$ converge to the values of the solution of boundary value problem (2.3) at the "shifted" points $(x - s_0(t) + s_0^h(t), t)$. Hence, we say that the function $z_{(5.21)}(x,t)$, $(x,t) \in \overline{G}_{h(5.21)}$ converges to the exact solution (ε, δ)-uniformly up to the shift of the argument x on the value of $s_0(t) - s_0^h(t)$, i.e., on the distance between the exact and numerical trace of the local disturbance (or, shortly, up to the shift of the argument). The quantity $s_0(t) - s_0^h(t)$, that is, the shift of the argument, converges to zero (ε, δ)-uniformly as N, $N_0 \to \infty$.

THEOREM 6. *Let the solution of boundary value problem* (2.3) *satisfy the estimates of Theorem* 2. *Then the solution of finite difference scheme* (5.1), (5.3), (4.2), (5.17), (5.19), (5.20) *converges to the solution of the boundary value problem (ε, δ)-uniformly up to the shift of the argument. The solution of the difference scheme and the disturbance trace $s_0(t)$ satisfy estimates* (5.22) *and* (5.15).

REMARK. In the case of problem (2.3), when the components $V(x,t)$ in the representation (3.12) satisfies estimate (3.11a), the accuracy the approximate solution may be rather improved. In this case we define the function $z_{((5.23))}(x,t)$, $(x,t) \in \overline{G}_{h((5.23))}$, assuming

$$\begin{gathered} z(x,t) = z_{(5.3,4.2,5.17,5.19)}(x,t), \\ (x,t) \in \overline{G}_{h(5.23)}, \overline{G}_{h(5.23)} = \overline{G}^h_{h(5.19)} \quad \text{for } \varepsilon \le \delta, \\ z(x,t) = z_{(5.3,4.2,5.17,5.20)}(x,t), \\ (x,t) \in \overline{G}_{h(5.23)}, \overline{G}_{h(5.23)} = \overline{G}^h_{h(5.20)} \quad \text{for } \varepsilon > \delta. \end{gathered} \tag{5.23}$$

For the function $z_{(5.23)}(x,t)$, $(x,t) \in \overline{G}_{h(5.23)}$, which is the solution of difference scheme (5.3), (4.2), (5.17), (5.20), we obtain the estimates

$$\left|u(x,t) - z_{(5.23)}(x + s_0(t) - s_0^h(t), t)\right| \le \begin{cases} M[N^{-1}\ln N + N_0^{-1}], & \varepsilon \le \delta, \\ M[N^{-1/3} + N_0^{-1/5}], & \varepsilon > \delta, \end{cases}$$
$$\text{for } \; |\, x - s_0^h(t)\,| \le m, \quad (x,t) \in \overline{G}_h;$$

$$\left|u(x,t) - z_{(5.23)}(x,t)\right| \le \begin{cases} M\,[\,N^{-1}\ln N + N_0^{-1}], & \varepsilon \le \delta, \\ M\,[\,N^{-1/3} + N_0^{-1/5}], & \varepsilon > \delta, \end{cases}$$
$$\text{for } \; |\, x_1 - s_0^h(t)\,| \ge m, \quad (x,t) \in \overline{G}_h, \quad \overline{G}_h = \overline{G}_{h(5.23)}.$$

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